SECTION DE MATHÉMATIQUES

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FUNCTIONAL ANALYSIS 2
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SHEET 6

Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 6.0 (True-or-false : a recap of the first weeks)
Decide for each of the statements if they are always true or if there are counterexamples. Justify your answers.

a) Let X be a vector space and $A, B \subset X$ be absorbing, balanced, convex sets such that $A \subset B$. Then the associated Minkowski-functionals satisfy $p_B \leq p_A$.
b) Let X be a vector space and $(A_i)_{i \in I}$ be a family of absorbing sets in X. Then $\cap_{i \in I} A_i$ is absorbing.
c) Let X be a TVS and d be a metric that generates the topology on X. If $d(x_n, x_m) \to 0$ as $n, m \to +\infty$, then $(x_n)_{n \in \mathbb{N}}$ is a (topological) Cauchy sequence in X.
d) The geometric version of the Hahn–Banach Theorem holds in every metrizable TVS.
e) A Hausdorff TVS that satisfies the Heine–Borel property is finite dimensional.
f) Every metrizable TVS that satisfies the Heine–Borel property is finite dimensional.
g) The space $\mathcal{D}_K$ is infinite dimensional if and only if K has non-empty interior.
h) Let X, Y be TVS and $T : X \to Y$ be linear and continuous. Then T maps bounded sets to bounded sets.

Ex 6.1 (The topology induced by a TVS on a linear subspace)
Let (X, τ_X) be a TVS and Y be a linear subspace. Endow Y with the topology τ_Y with the topology induced by X ; namely, open sets $\tilde{U} \in \tau_Y$ are all those if the form $\tilde{U} = U \cap Y$, for some $U \in \tau_X$. Show the following :

i) A sequence $(y_n)_{n \in \mathbb{N}} \subset Y$ converges to $y \in Y$ w.r.t. τ_Y iff it converges to y w.r.t. τ_X.
ii) (Y, τ_Y) is a TVS.
iii) A sequence $(y_n)_{n \in \mathbb{N}} \subset Y$ is Cauchy w.r.t. τ_Y iff it is Cauchy w.r.t. τ_X.
iv) If (X, τ_X) is a LCTVS, then so is (Y, τ_Y).
v) A set $E \subset Y$ is bounded w.r.t. τ_Y iff it is bounded w.r.t. τ_X.
vi) A set $E \subset Y$ is compact w.r.t. τ_Y iff it is compact w.r.t. τ_X.
vii) If (X, τ_X) is topologically complete and Y is closed w.r.t. τ_X, then (Y, τ_Y) is topologically complete.
viii) If (X, τ_X) is a Fréchet space and Y is closed w.r.t. τ_X, then (Y, τ_Y) is Fréchet.
ix) If (X, τ_X) has the Heine–Borel property and Y is closed w.r.t. τ_X, then (Y, τ_Y) has the Heine–Borel property.

Ex 6.2 (Continuous functions on $\mathcal{D}(\Omega)$)
a) Show that the following linear maps are continuous from $\mathcal{D}(\Omega)$ to itself :

i) $\varphi \mapsto D^\alpha \varphi$ for $\alpha \in \mathbb{N}_0^d$;
ii) $\varphi \mapsto \psi\varphi$ for $\psi \in C^\infty(\Omega)$;

iii) $\varphi \mapsto \{x \mapsto \varphi(\lambda x - z)\}$ with $\lambda \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{R}^d$ fixed (for $\Omega = \mathbb{R}^d$)

b) Show that the inclusion $\iota : \mathcal{D}(\Omega) \to C^\infty(\Omega)$ is continuous, where $C^\infty(\Omega)$ is equipped with the topology inherited from the seminorms $(p_N)_{N\in\mathbb{N}}$ given in Definition 2.3.

c) Let $\alpha \in \mathbb{N}_0^d$ and μ be a Borel-measure [1] that is finite on compact sets of Ω. Show that the linear functional

$$G(\varphi) = \int_\Omega D^\alpha \varphi(x) \, \mathrm{d}\mu(x)$$

is continuous on $\mathcal{D}(\Omega)$.
Hint: You may use the following result : given a LCTVS Y, a linear map $T : \mathcal{D}(\Omega) \to Y$ is continuous if and only if it is sequentially continuous in the origin ; cf. Prop. 2.13.

Ex 6.3 (Non-metrizability of $\mathcal{D}(\Omega)$)
Let $\Omega \subset \mathbb{R}^d$ be open and let τ be the topology on $\mathcal{D}(\Omega)$ given by Definition 2.7. Show that this topology is not metrizable.
Hint: Recall the Baire category theorem in the following version : Let (X, d) be a complete metric space, then any countable union of closed sets with empty interior has empty interior.

Ex 6.4 (The Fourier transformation on $\mathcal{D}(\Omega)$ *)
Let $\Omega \subset \mathbb{R}^d$ be open and $\varphi \in \mathcal{D}(\Omega)$.

a) Show that $\varphi \in L^p(\mathbb{R}^d)$ for all $1 \leq p \leq +\infty$.

b) By a), the Fourier transform

$$\mathcal{F}[\varphi](k) := \int_{\mathbb{R}^d} e^{-ik\cdot x} \varphi(x) \, \mathrm{d}x$$

is well-defined. Show that $\mathcal{F}[\varphi]$ belongs to $\mathcal{D}(\mathbb{R}^d)$ if and only if $\varphi \equiv 0$.
Hint: Consider first the case $d = 1$. Define the Fourier transform also for complex arguments and show that this function is holomorphic on $\mathbb{C}$. Then recall the identity theorem for analytic functions. When $d > 1$, fix $d - 1$ variables and repeat the one-dimensional strategy.

1. i.e. a measure defined on the Borel σ-algebra on Ω ; in case you haven't discussed general measure theory in your analysis courses, assume that $\mathrm{d}\mu(x) = f(x) \, \mathrm{d}x$ with f locally integrable.